Non-Trivial: Question 2 - Balancing Risk and Ambition

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1 The Problem

Whilst looking for research opportunities as an aspiring physicist, I stumbled upon *Non-Trivial*, a non-profit which supports young people who want to make an altruistic difference with their scientific talents.
I decided that I would like to apply too, and after submitting some personal information, I was given some very challenging brain-teasers! Of these, I was particularly intrigued by Question 2:
You are playing a game where the incentive is to maximise your score.
You play the game with four other randomly selected players.
You must pick a number, X, and your score is calculated using the following formula:

$$\text{Score} = X \cdot (50 - \sum_{i=1}^{4}(X_i) - X) \tag{1}$$

Where X_1, X_2, X_3 and X_4 are the number picked by the other players.

2 Solutions

2.1 The No-Risk Solution

Consider the following scenarios:

1. If $\sum_{i=1}^{4}(X_i)$, the sum of everyone else's numbers, is a very large positive number then $(50 - \sum_{i=1}^{4}(X_i) - X)$ becomes a very large negative number. This means that we could pick a negative number which would allow us to get a high score. However, choosing too large a number risks making $(50 - \sum_{i=1}^{4}(X_i) - X)$ positive, thus giving us a negative score.

2. Similarly, If $\sum_{i=1}^{4}(X_i)$ is a very large negative number, $(50 - \sum_{i=1}^{4}(X_i) - X)$ becomes a very large positive number. Thus, we could pick a positive number to get a high score, but if our number is too big, $(50 - \sum_{i=1}^{4}(X_i) - X)$ turns negative and we risk having a negative score yet again.

Using graphical methods will show you the same thing, that is: either you will always get a negative score or it will peak at a point and then collapse to negativity. Therefore, because the probability of having a negative number is greater than the probability of having a positive number, the best answer using this logic should be **0**, as this ensures that we never get a negative number.

2.2 The Altruistic Approach

Considering that this problem was for an application for an altruistic company, then we should all demonstrate our altruism by taking a risk and trying to maximise all of our scores.
If we all pick the same number, X, then our scores become $X * (50 - 5X)$, which is a simple quadratic $(-5X^2 + 50X)$. This has an easily calculable maximum at $(5, 125)$, which means that we should all pick **5** to give us all a score of 125.

2.3 Considering All Possibilities Using Sums

Let's start by defining a function $f(x, y)$ which represents our score. Here, x represents the number we chose and $y = \sum_{i=1}^{4}(X_i)$, the sum of everyone else's numbers. Thus, using the information given to us in the question, we can deduce:

$$f(x, y) = x \cdot (50 - x - y) \tag{2}$$

We can set a constant y and see how varying x affects our score. For example, setting $y = 5$ will give us $f(x, 5)$, which is $x \cdot (50 - x - 5)$. No matter what we change y to, we should always get a graph that is a negative parabola.
But we want to consider all the numbers everyone else chooses - people are random and can have some pretty wild choices!
The way we would do this is by summing up all of the functions at different values of y (e.g. $(f(x, -2) + f(x, -1) + f(x, 0) + f(x, 1) + f(x, 2) + ...)$). This would give us a function $g(x)$ whose maximum (if it has one) would be the optimal solution, as it would take into account all the functions.
Let's say we want g for now to only consider the functions where y is an integer and $-10 \leq y \leq 10$. We can define g as:

$$g(x) = \sum_{i=-10}^{10} f(x, i) = \sum_{i=-10}^{10} (x \cdot (50 - x - i)) \tag{3}$$

Notice that the lower and upper bounds of y are the lower and upper bounds of the summation. Thus, we can deduce that for $y : y \in \mathbf{Z}, -u \leq y \leq u$, g is:

$$g(x) = \sum_{i=-u}^{u} f(x, i) = \sum_{i=-u}^{u} (x \cdot (50 - x - i)) \tag{4}$$

Putting this into a professional tool such as Wolfram Alpha, Equation 4 becomes:

$$g(x) = -x \cdot ((x - 50) \cdot (2u + 1)) \tag{5}$$

We see that for any (positive) u, there is a maximum at $x = 25$, even as u approaches infinity ($\lim_{u \to \infty}$). Thus, we can at last conclude that the optimal solution must be **25**.

3 My Judgement

I ended up picking **5** because I feel it was a real way to demonstrate my altruism. Please note that this entire document was written in a day (although the thought process took much longer), thus there may be some mistakes (especially with my logic in Section 2.3). With that said, thank you for reading!